

# AN EXAMPLE OF A NON-BOREL LOCALLY-CONNECTED FINITE-DIMENSIONAL TOPOLOGICAL GROUP

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**ABSTRACT.** Answering a question posed by S.Maillot in MathOverflow, for every  $n \in \mathbb{N}$  we construct a locally connected subgroup  $G \subset \mathbb{R}^{n+1}$  of dimension  $\dim(G) = n$ , which is not locally compact.

By a classical result of Gleason [3] and Montgomery [6], every locally path-connected finite-dimensional topological group  $G$  is locally compact. Generalizing this result of Gleason and Montgomery, Banakh and Zdomskyy [1] proved that a topological group  $G$  is locally compact if  $G$  is compactly finite-dimensional and locally continuum-connected. In [5] Sylvain Maillot asked if the locally path-connectedness in the result of Gleason and Montgomery can be replaced by the local connectedness. In this paper we construct a counterexample to this question of S. Maillot. We recall that a subset  $B$  of a Polish space  $X$  is called a *Bernstein set* in  $X$  if both  $B$  and  $X \setminus B$  meet every uncountable closed subset  $F$  of  $X$ . Bernstein sets in Polish space can be easily constructed by transfinite induction, see [4, 8.24].

**Theorem 1.** *For every  $n \geq 1$  the Euclidean space  $\mathbb{R}^n$  contains a dense additive subgroup  $G \subset \mathbb{R}^n$  such that*

- (1)  $G$  is a Bernstein set in  $\mathbb{R}^n$ ;
- (2)  $G$  is locally connected;
- (3)  $G$  has dimension  $\dim(G) = n - 1$ ;
- (4)  $G$  is not Borel and hence not locally compact.

*Proof.* Let  $(F_\alpha)_{\alpha < \mathfrak{c}}$  be an enumeration of all uncountable closed subsets of  $\mathbb{R}^n$  by ordinal  $\alpha < \mathfrak{c}$ . Fix any point  $p \in \mathbb{R}^n \setminus \{0\}$ . By transfinite induction, for every ordinal  $\alpha < \mathfrak{c}$  we shall choose a point  $z_\alpha \in F_\alpha$  such that the subgroup  $G_\alpha \subset \mathbb{R}^n$  generated by the set  $\{z_\beta\}_{\beta < \alpha}$  does not contain the point  $p$ . Assume that for some ordinal  $\alpha < \mathfrak{c}$  we have chosen points  $z_\beta$ ,  $\beta < \alpha$ , so that the subgroup  $G_{<\alpha}$  generated by the set  $\{z_\beta\}_{\beta < \alpha}$  does not contain  $p$ . Consider the set  $Z = \{\frac{1}{n}(p - g) : n \in \mathbb{Z} \setminus \{0\}, g \in G_{<\alpha}\}$  and observe that it has cardinality  $|Z| \leq \omega \cdot |G_{<\alpha}| \leq \omega + |\alpha| < \mathfrak{c}$ . Since the uncountable closed subset  $F_\alpha$  of  $\mathbb{R}^n$  has cardinality  $|F_\alpha| = \mathfrak{c}$  (see [4, 6.5]), there is a point  $z_\alpha \in F_\alpha \setminus Z$ . For this point we get  $p \neq nz_\alpha + g$  for any  $n \in \mathbb{Z} \setminus \{0\}$ , and  $g \in G_{<\alpha}$ . Consequently, the subgroup  $G_\alpha = \{nz_\alpha + g : n \in \mathbb{Z}, g \in G_{<\alpha}\}$  generated by the set  $\{z_\beta\}_{\beta \leq \alpha}$  does not contain the point  $p$ . This completes the inductive step.

After completing the inductive construction, consider the subgroup  $G$  generated by the set  $\{a_\alpha\}_{\alpha < \mathfrak{c}}$  and observe that  $p \notin G$  and  $G$  meets every uncountable closed subset  $F$  of  $\mathbb{R}^n$ . Moreover, since  $G$  meets the closed uncountable set  $F - p$ , the coset  $p + G \subset \mathbb{R}^n \setminus G$  meets  $F$ . So, both the subgroup  $G$  and its complement  $\mathbb{R}^n \setminus G$  meet each uncountable closed subset of  $\mathbb{R}^n$ , which means that  $G$  is a Bernstein set in  $\mathbb{R}^n$ . The following proposition implies that the group  $G$  has properties (2)–(4).  $\square$

**Proposition 1.** *Let  $n \geq 2$ . Every Bernstein subset  $B$  of  $\mathbb{R}^n$  has the following properties:*

- (1)  *$B$  is not Borel;*
- (2)  *$B$  is connected and locally connected;*
- (3)  *$B$  has dimension  $\dim(B) = n - 1$ .*

*Proof.* 1. By [4, 8.24], the Bernstein set  $B$  is not Borel (more precisely,  $B$  does not have the Baire property in  $\mathbb{R}^n$ ).

2. To prove that  $B$  is connected and locally connected, it suffices to prove that for every open subset  $U \subset \mathbb{R}^n$  homeomorphic to  $\mathbb{R}^n$  the intersection  $U \cap B$  is connected. Assuming the opposite, we could find two non-empty open disjoint sets  $U_1, U_2 \subset U$  such that  $U \cap B = (U_1 \cap B) \cup (U_2 \cap B)$ . Consider the complement  $F = U \setminus (U_1 \cup U_2) \subset U \setminus B$  and observe that  $F$  is closed in  $U$  and hence of type  $F_\sigma$  in  $\mathbb{R}^n$ . If  $F$  is uncountable, then  $F$  contains an uncountable closed subset of  $\mathbb{R}^n$  and hence meets the set  $B$ , which is not the case. So, the closed subset  $F$  of  $U$  is at most countable and separates the space  $U \cong \mathbb{R}^n$ , which contradicts Theorem 1.8.14 of [2].

3. Since the subset  $B$  has empty interior in  $\mathbb{R}^n$ , we can apply Theorem 1.8.11 of [2] and conclude that  $\dim(B) < n$ . On the other hand, Lemma 1.8.16 [2] guarantees that  $B$  has dimension  $\dim(B) \geq n - 1$  (since  $B$  meets every non-trivial compact connected subset of  $\mathbb{R}^n$ ). So,  $\dim(B) = n - 1$ .  $\square$

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